On some drawbacks and possible improvements of a Lagrangian Finite Element approach for simulating incompressible flows

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My PhD focuses on the analysis and development of the PFEM for new applications involving free surfaces/interfaces

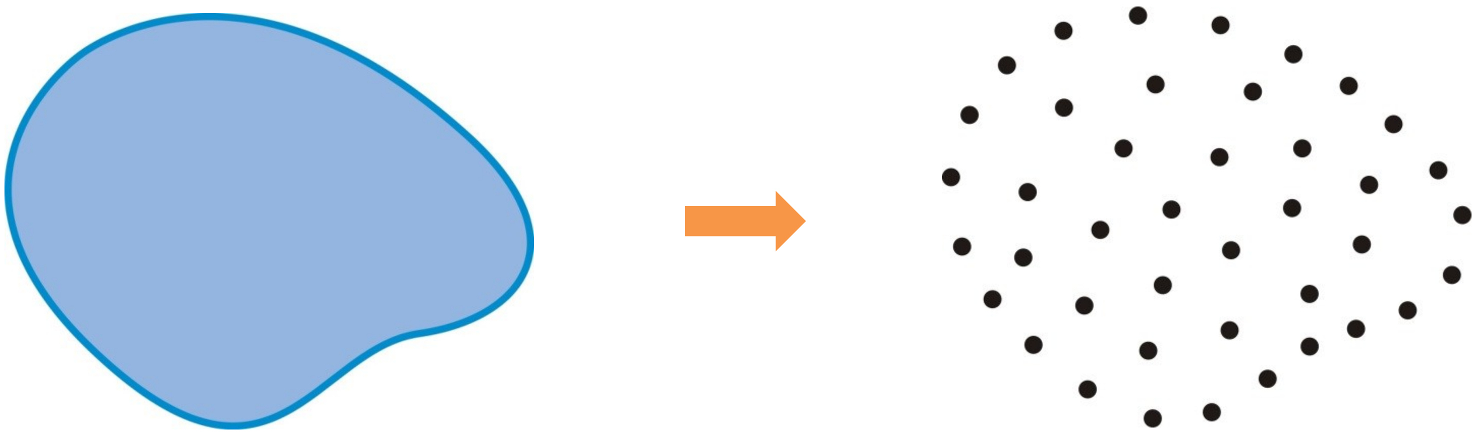
Bird strike on a wind shield test

Presentation layout

- PFEM general ideas
- Correct formulation for incompressible free-surface flows
- PFEM issues
- Conclusions

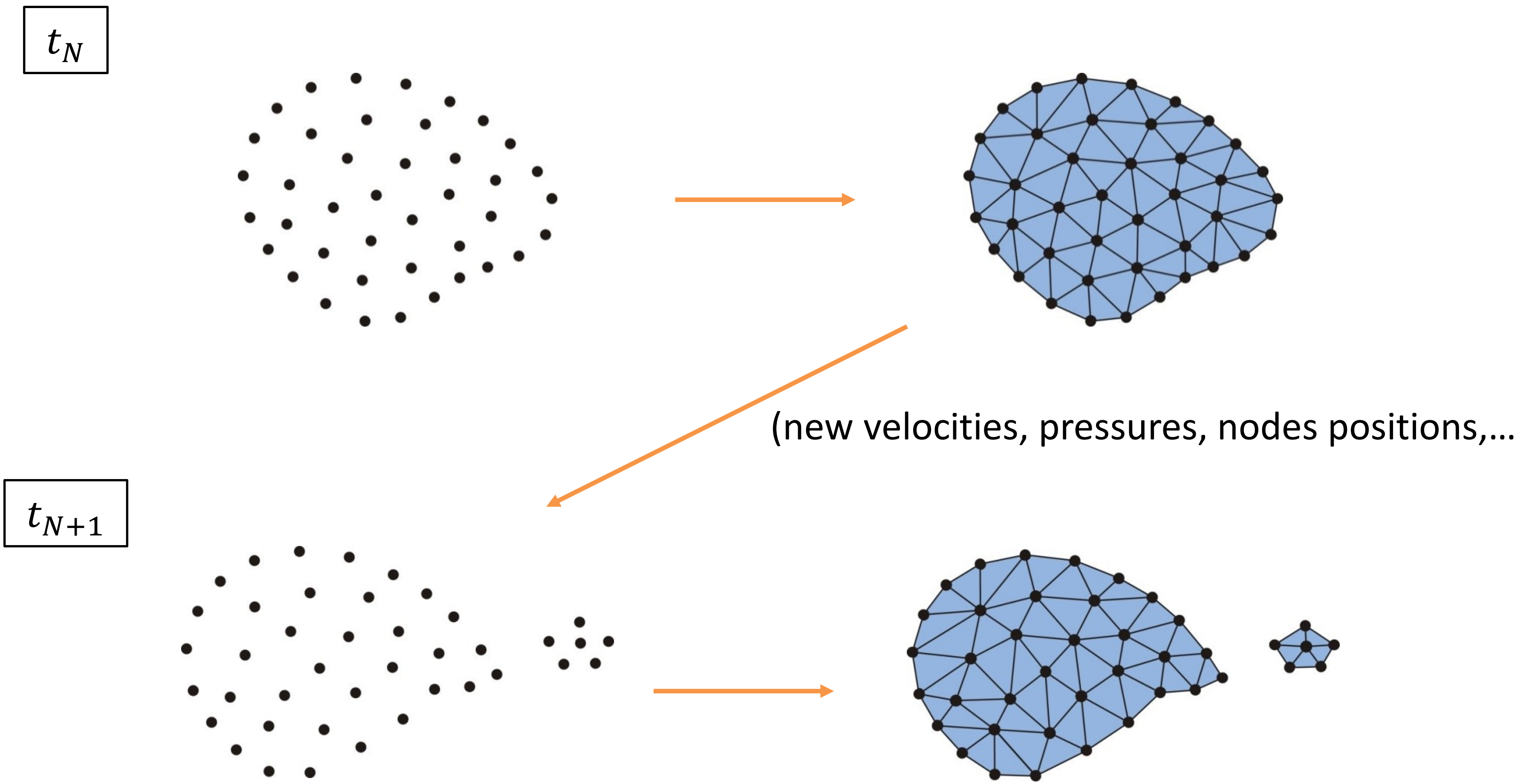
PFEM general ideas

The first step in the PFEM is discretizing the continuum with some particles/nodes



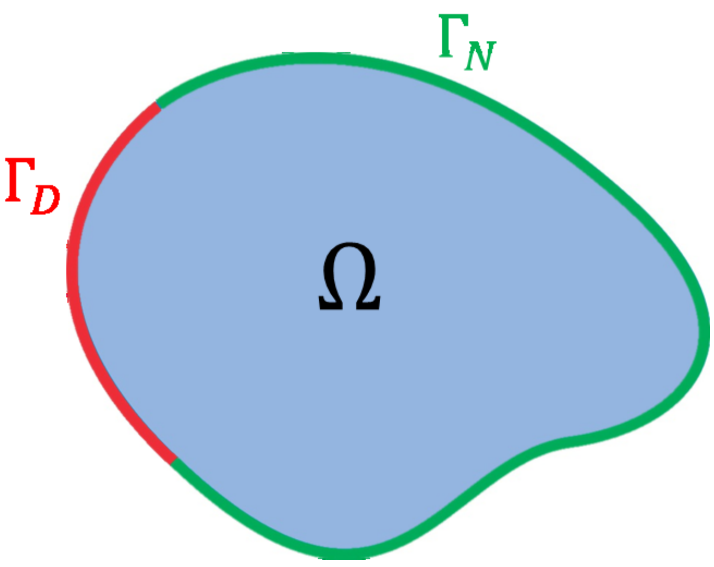
The particles carry all the physical and mathematical information (density, viscosity, velocity, pressure, …)

Then, particles are free to move and at each time step a new mesh is built in order to define the weak form



Formulation for incompressible free-surface flows

The starting point are the equations of the continuum written in Lagrangian form and current configuration



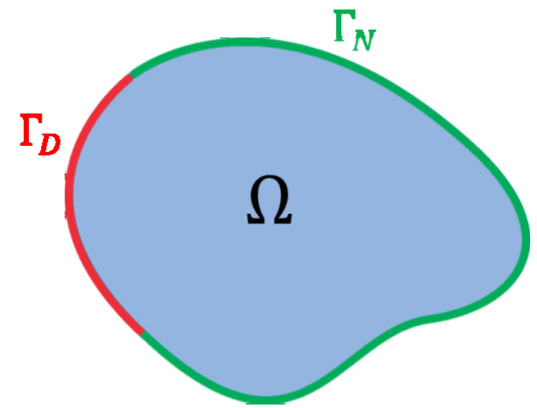
$$\begin{cases} \rho \dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} \quad \text{in } \boldsymbol{\Omega} \\ \dfrac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in } \boldsymbol{\Omega} \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}} \end{cases}$$

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x}, t) = \bar{\boldsymbol{u}}(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \Gamma_D \\ \boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot \boldsymbol{n} = \bar{\boldsymbol{t}}(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \Gamma_N \end{cases}$$

From now on I will focus on Newtonian incompressible fluid flows

$$\begin{cases} \rho \dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b} \quad \text{in } \boldsymbol{\Omega} \\ \dfrac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in } \boldsymbol{\Omega} \\ (\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}) \end{cases}$$

$$\begin{cases} \boldsymbol{u}(\boldsymbol{x}, t) = \bar{\boldsymbol{u}}(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \Gamma_D \\ \boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot \boldsymbol{n} = \bar{\boldsymbol{t}}(\boldsymbol{x}, t) \quad \forall \boldsymbol{x} \in \Gamma_N \end{cases}$$



$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\mu \mathbf{D}(\boldsymbol{u}) \,, \quad \mathbf{D}(\boldsymbol{u}) = \frac{1}{2}\left(\operatorname{grad}(\boldsymbol{u}) + \operatorname{grad}(\boldsymbol{u})^{\mathrm{T}}\right)$$

$$\Rightarrow \begin{cases} \rho_0 \dfrac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\operatorname{div}(p\mathbf{I}) + \mu \operatorname{div}\left(\operatorname{grad}(\boldsymbol{u}) + \operatorname{grad}(\boldsymbol{u})^{\mathrm{T}}\right) + \rho_0 \boldsymbol{b} \quad \text{in } \boldsymbol{\Omega} \\ \operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in } \boldsymbol{\Omega} \end{cases}$$

A stable weak form can be obtained by using a Galerkin approach and a Petrov-Galerkin stabilization for pressure

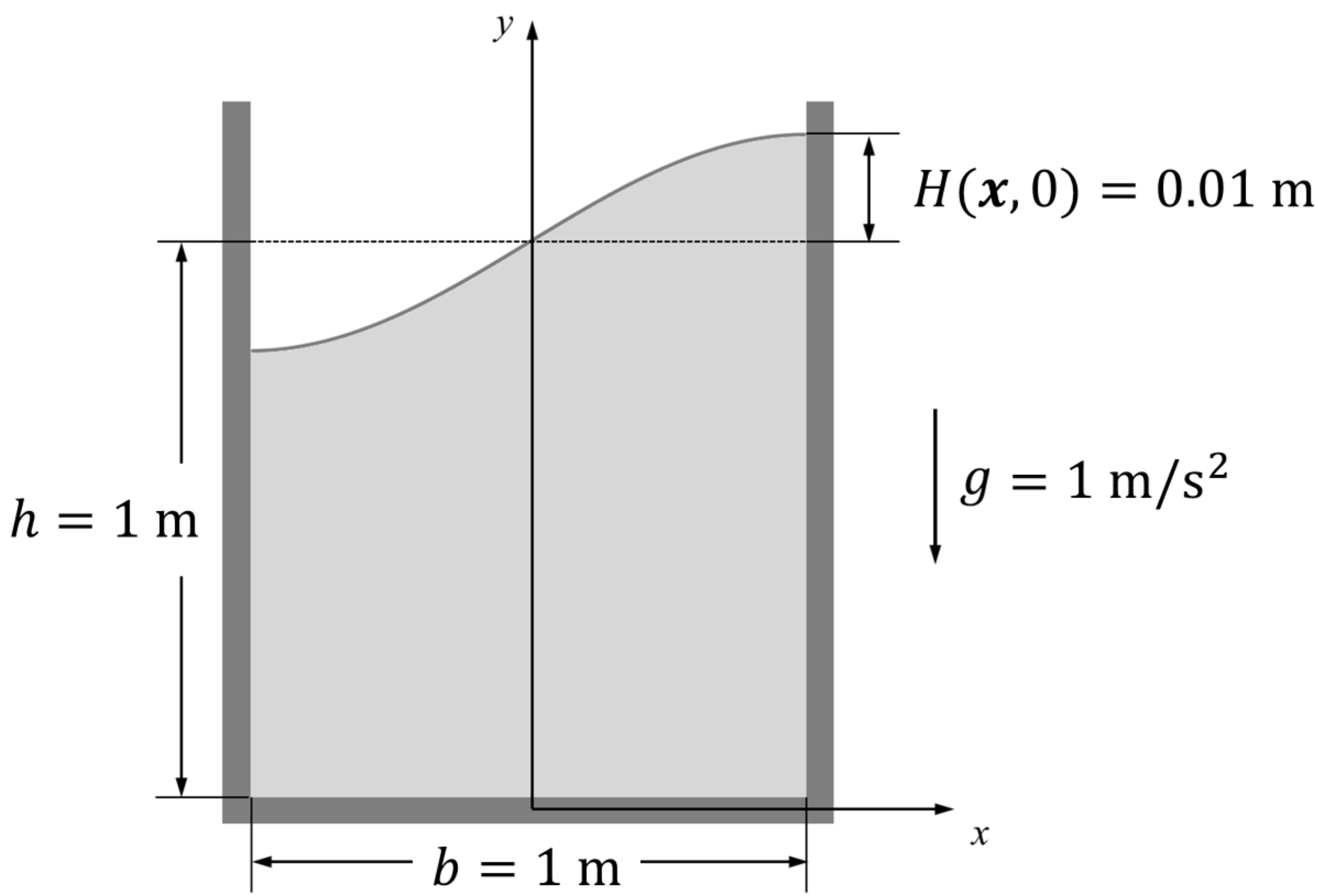
$$\begin{cases} \displaystyle\int_\Omega \rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \boldsymbol{w} \, \mathrm{d}\Omega = \int_\Omega p\mathbf{I} : \mathrm{grad}(\boldsymbol{w}) \, \mathrm{d}\Omega - \int_\Omega \mu \, \mathrm{grad}(\boldsymbol{u}) : \mathrm{grad}(\boldsymbol{w}) \, \mathrm{d}\Omega \; + \\ \qquad\qquad - \displaystyle\int_\Omega \mu \, \mathrm{grad}(\boldsymbol{u})^{\mathrm{T}} : \mathrm{grad}(\boldsymbol{w}) \, \mathrm{d}\Omega + \int_\Omega \rho_0 \, \boldsymbol{b} \cdot \boldsymbol{w} \, \mathrm{d}\Omega + \int_{\Gamma_N} \bar{\boldsymbol{t}} \cdot \boldsymbol{w} \, \mathrm{d}\Gamma \\ \displaystyle\int_\Omega \mathrm{div}(\boldsymbol{u}) q \, \mathrm{d}\Omega + \sum_{e=1}^{N_{el}} \int_{\Omega_0^e} \tau_{\mathrm{pspg}}^e \frac{1}{\rho_0} \mathrm{grad}(q) \left(\rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \mathrm{div}(p\mathbf{I}) - \mu \, \mathrm{div}\left(\mathrm{grad}(\boldsymbol{u}) + \mathrm{grad}(\boldsymbol{u})^{\mathrm{T}}\right) - \rho_0 \boldsymbol{b} \right) \end{cases}$$

[Tezduyar *et al.* (1992), Cremonesi *et al*. (2010)]

$$\forall \boldsymbol{w} \in \boldsymbol{H}^1(\Omega) \,|\, \boldsymbol{w} = \boldsymbol{0} \text{ on } \Gamma_D, \qquad \forall q \in L^2(\Omega)$$

$$\begin{cases} \mathbf{M} \dfrac{\boldsymbol{u}^{\mathrm{n}+1} - \boldsymbol{u}^{\mathrm{n}}}{\Delta t} + \mathbf{K}\boldsymbol{u} + \mathbf{D}^{\mathrm{T}} \boldsymbol{p} = \boldsymbol{B} \\ \mathbf{C} \dfrac{\boldsymbol{u}^{\mathrm{n}+1} - \boldsymbol{u}^{\mathrm{n}}}{\Delta t} + \mathbf{D}\boldsymbol{u} + \mathbf{L}\boldsymbol{p} = \boldsymbol{H} \end{cases}$$

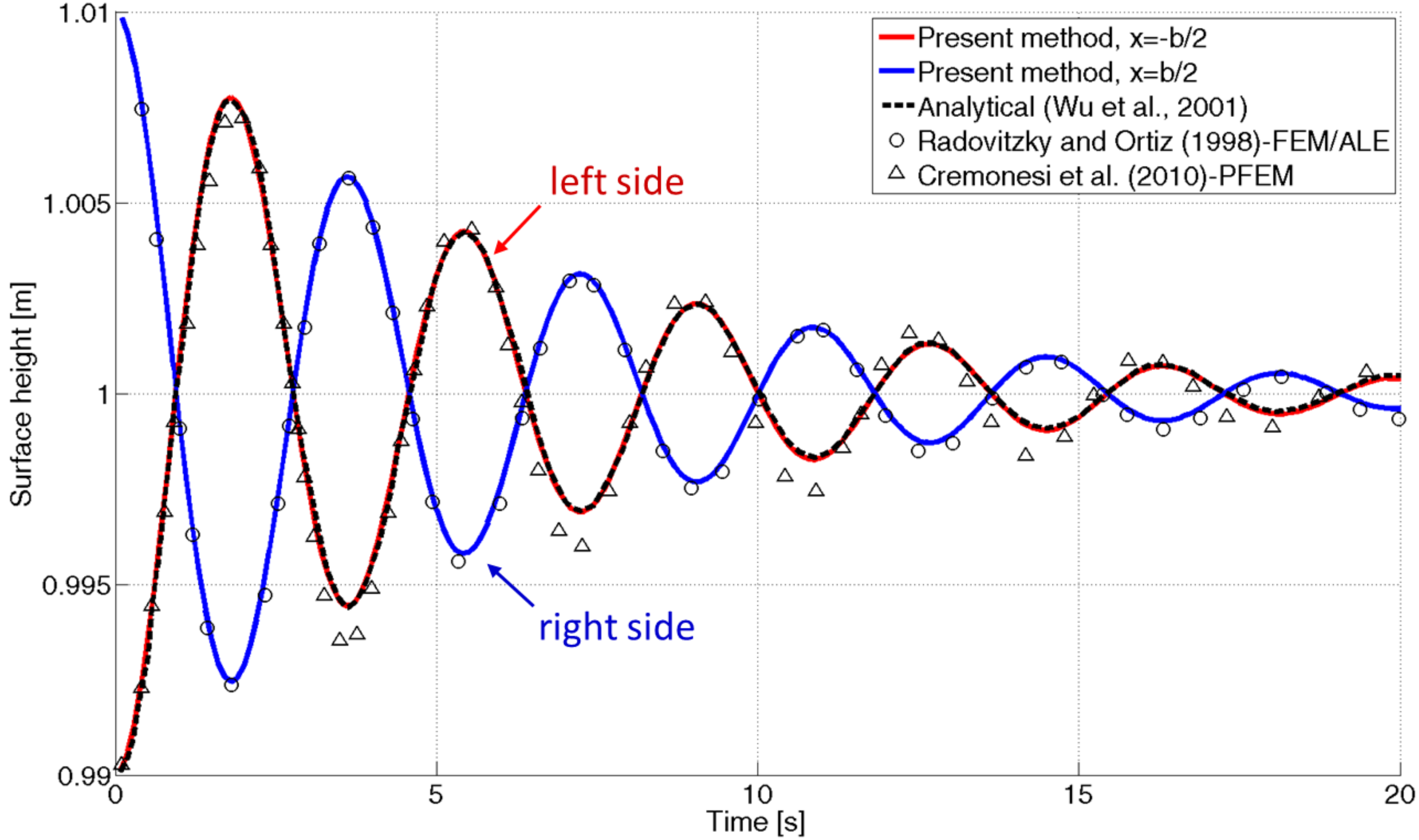
The method has been validated against an analytical solution for the free-surface evolution of a classical sloshing example



$\rho_0 = 1\ \mathrm{kg/m^3}$

$\mu = 0.01\ \mathrm{kg/ms}$

Our results perfectly agree with the analytical solution but show some differences with those found by other authors



For free-surface flows some dangerous simplifications are often proposed in the literature

1. Strong imposition of the pressure at the free surface

$$\int_\Omega \rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \boldsymbol{w}\, \mathrm{d}\Omega \quad = (\dots) + \underbrace{\cancel{\int_{\Gamma_N} \bar{\boldsymbol{t}} \cdot \boldsymbol{w}\, \mathrm{d}\Gamma}}_{\text{neglected}} \quad \Rightarrow \quad p = 0\,, \qquad \text{on } \Gamma_\mathrm{N}$$

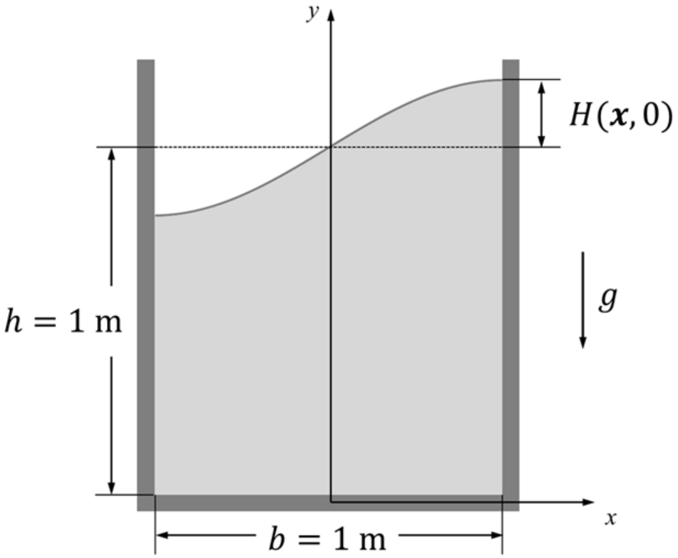
2. Wrong definition of the boundary term

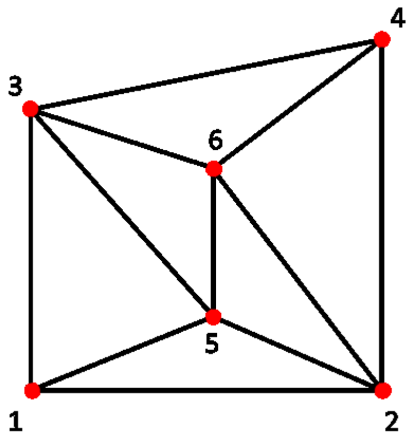
$$\mu\, \mathrm{div}\big(\mathrm{grad}(\boldsymbol{u}) + \mathrm{grad}(\boldsymbol{u})^\mathrm{T}\big) = \mu\, \Delta(\boldsymbol{u}), \qquad \text{for incompressible flows}$$

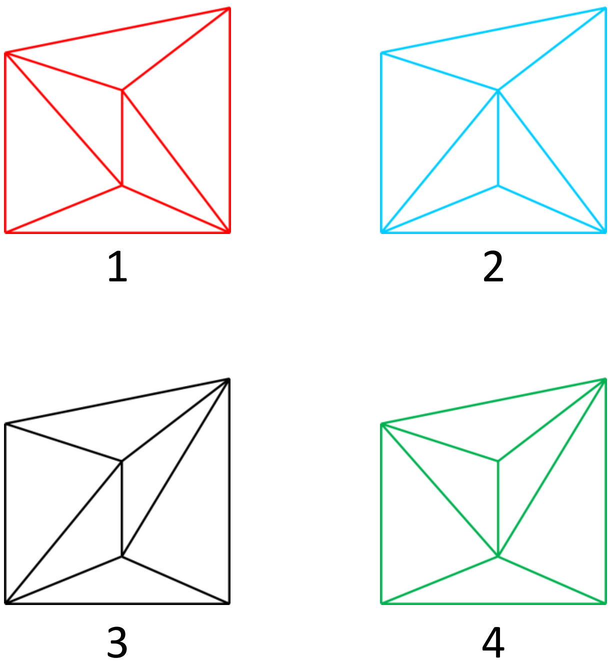
$$\int_\Omega \rho_0 \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \boldsymbol{w}\, \mathrm{d}\Omega \quad = (\dots) - \int_\Omega \mu\, \mathrm{grad}(\boldsymbol{u}) : \mathrm{grad}(\boldsymbol{w})\, \mathrm{d}\Omega \quad + \int_{\Gamma_N} (\, \underbrace{\bar{\boldsymbol{t}}}_{\text{« pseudo-tractions »}} - \underbrace{\cancel{\mu\, \mathrm{grad}(\boldsymbol{u})^\mathrm{T} \boldsymbol{n}}}_{\text{neglected}}\,) \cdot \boldsymbol{w}\, \mathrm{d}\Gamma$$

PFEM issues

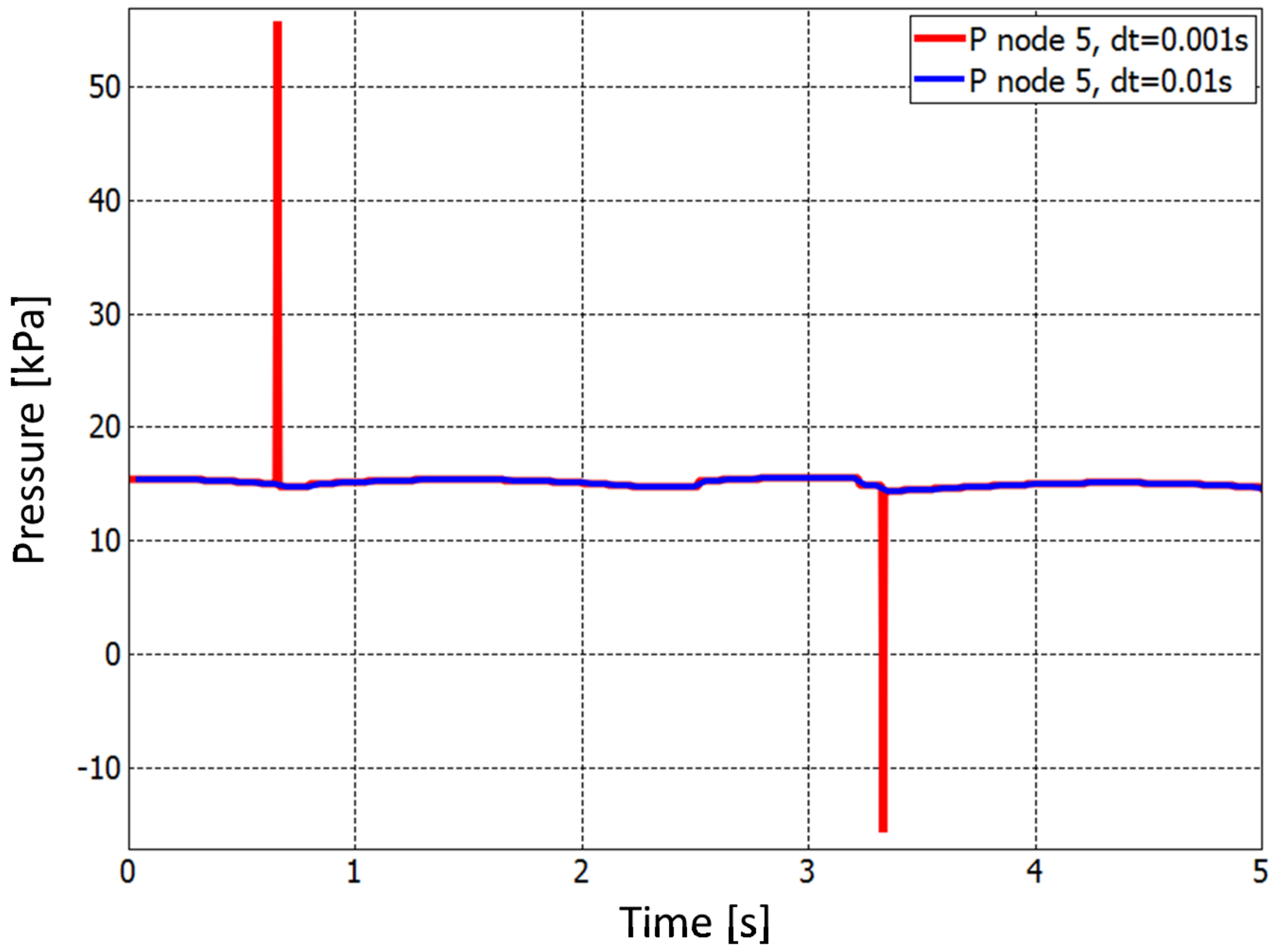
To introduce the problem, let’s consider again a sloshing example, but with a very coarse discretization

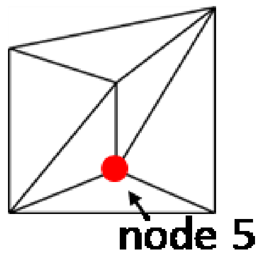




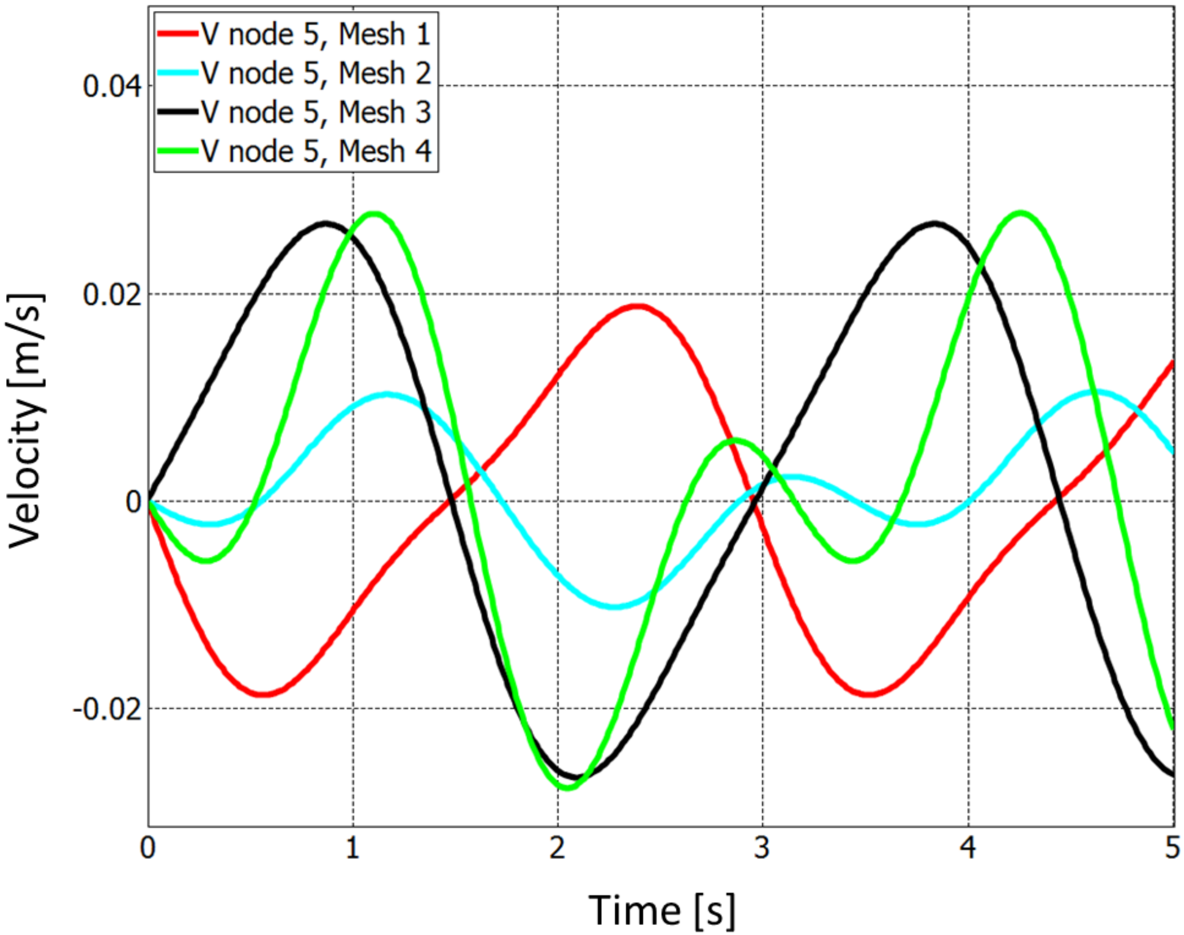


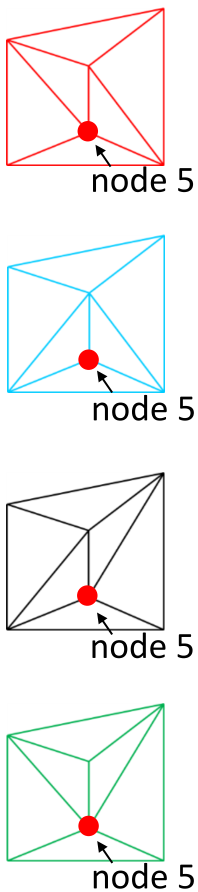
Some odd oscillations in the pressure field appear, at node 5 for instance, when the time step is « too » small



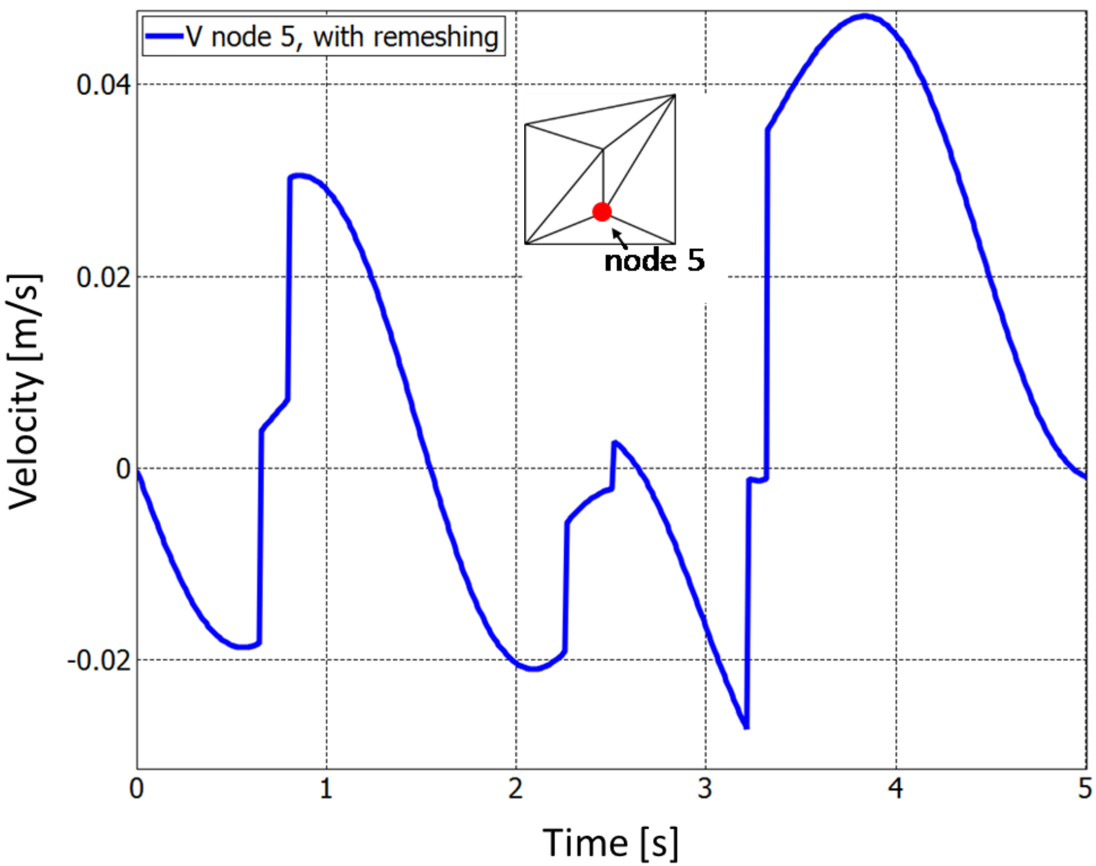


A first observation: the evolutions of the vertical velocity at node 5 for meshes 1 – 4, without performing any remeshing, are very different





The remeshing introduces perturbations in the velocity field which have to be counter-balanced by the pressure gradient



$$\boldsymbol{u}^{\mathrm{n+1}} = \overline{\boldsymbol{u}}^{\mathrm{n+1}} + \delta\boldsymbol{u}$$

$$\boldsymbol{p}^{\mathrm{n+1}} = \overline{\boldsymbol{p}}^{\mathrm{n+1}} + \delta\boldsymbol{p}$$

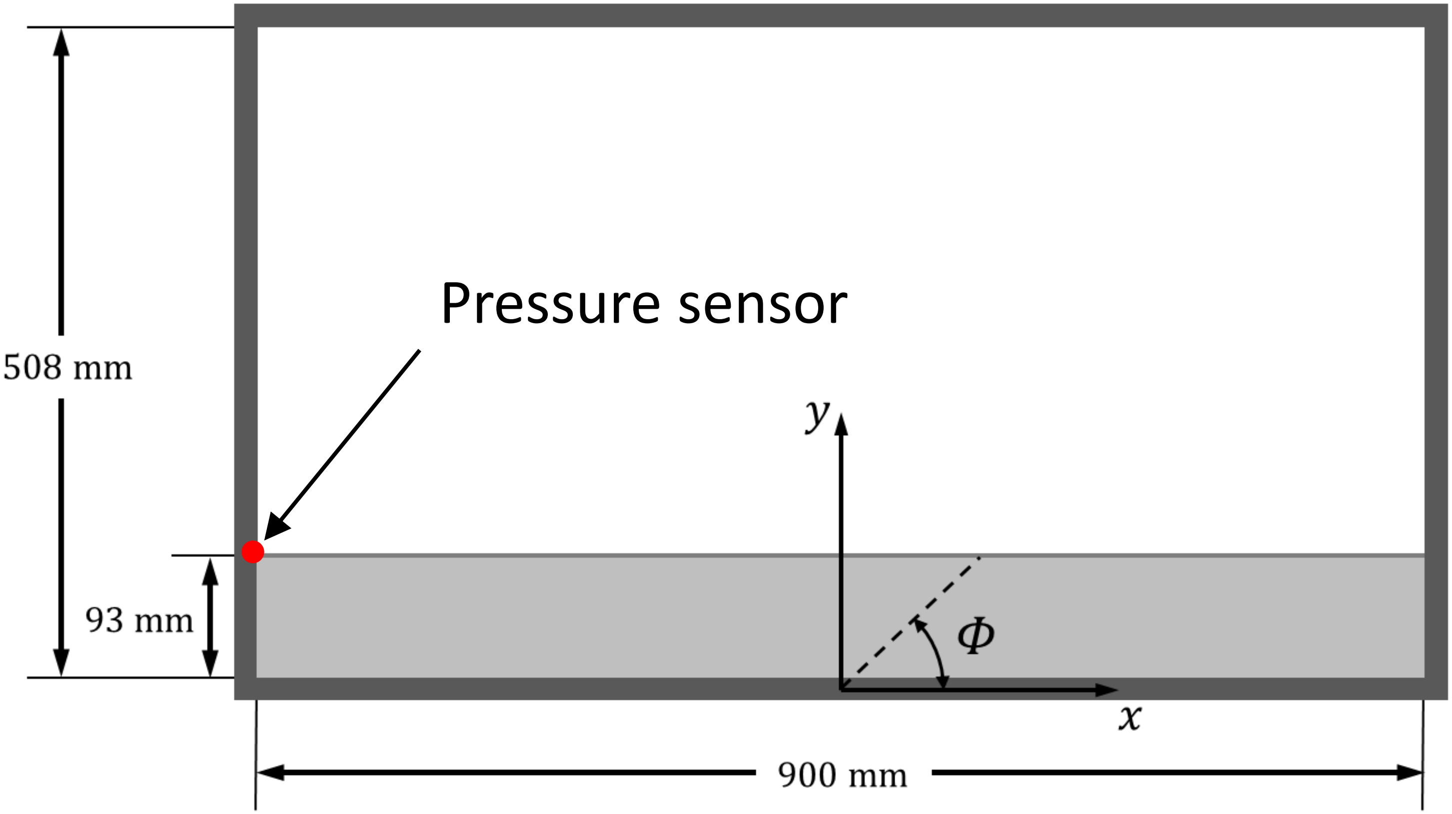
<u>Momentum balance:</u>

$$\mathbf{M}\frac{\overline{\boldsymbol{u}}^{\mathrm{n+1}} + \delta\boldsymbol{u} - \overline{\boldsymbol{u}}^{\mathrm{n}}}{\Delta t} + \mathbf{K}(\overline{\boldsymbol{u}}^{\mathrm{n+1}} + \delta\boldsymbol{u}) + \mathbf{D}^{\mathrm{T}}(\overline{\boldsymbol{p}}^{\mathrm{n+1}} + \delta\boldsymbol{p}) = \overline{\boldsymbol{B}}$$

$$\mathbf{M}\frac{\delta\boldsymbol{u}}{\Delta t} + \mathbf{K}\delta\boldsymbol{u} + \mathbf{D}^{\mathrm{T}}\delta\boldsymbol{p} = \mathbf{0}$$

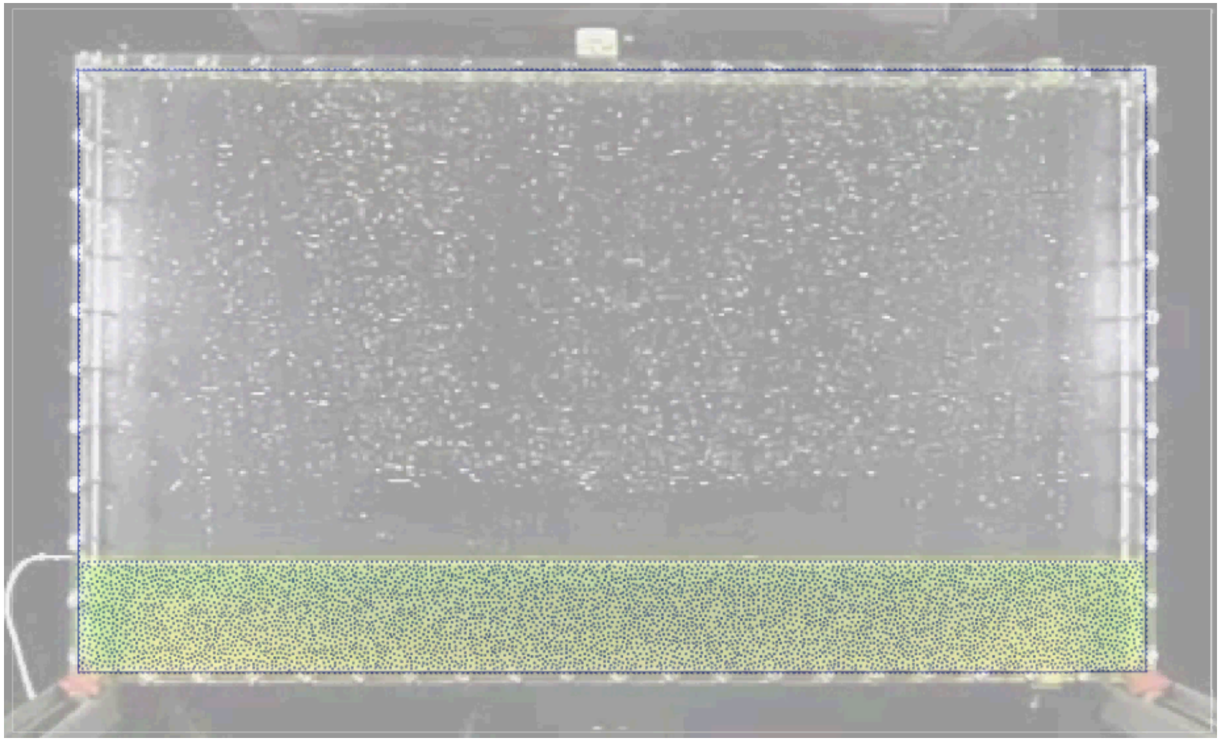
$$\delta\boldsymbol{p} = -\mathbf{D}^{-\mathrm{T}}\left[\frac{1}{\Delta t}\mathbf{M} + \mathbf{K}\right]\delta\boldsymbol{u}$$

To analyze these effects on a more realistic problem we consider the sloshing of an oscillating water reservoir



[Experimental results available online on the SPHERIC community website: https://wiki.manchester.ac.uk/spheric]

The present method can reproduce the global evolution of the phenomenon with very good accuracy



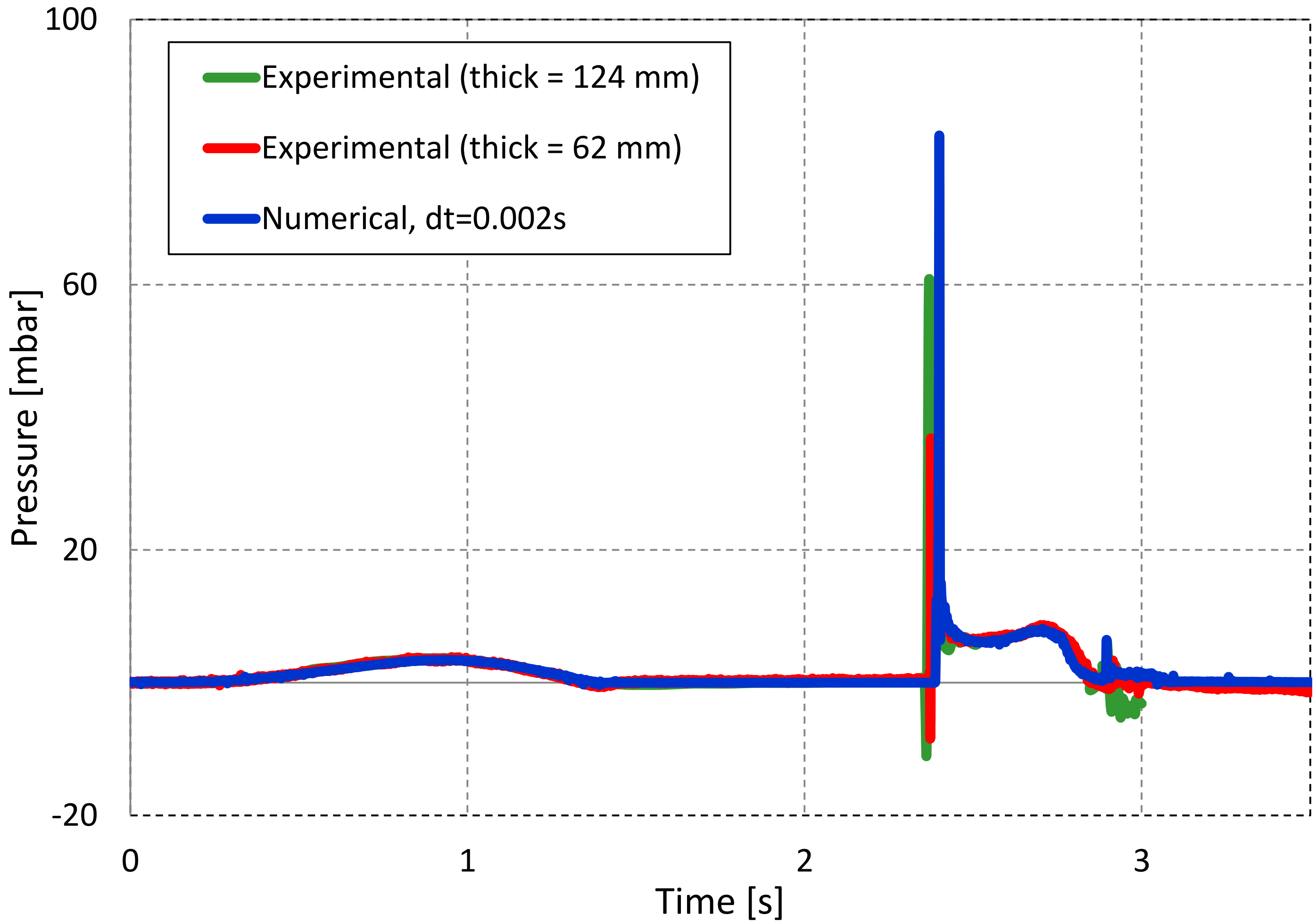
3.5s simulation

6000 particles

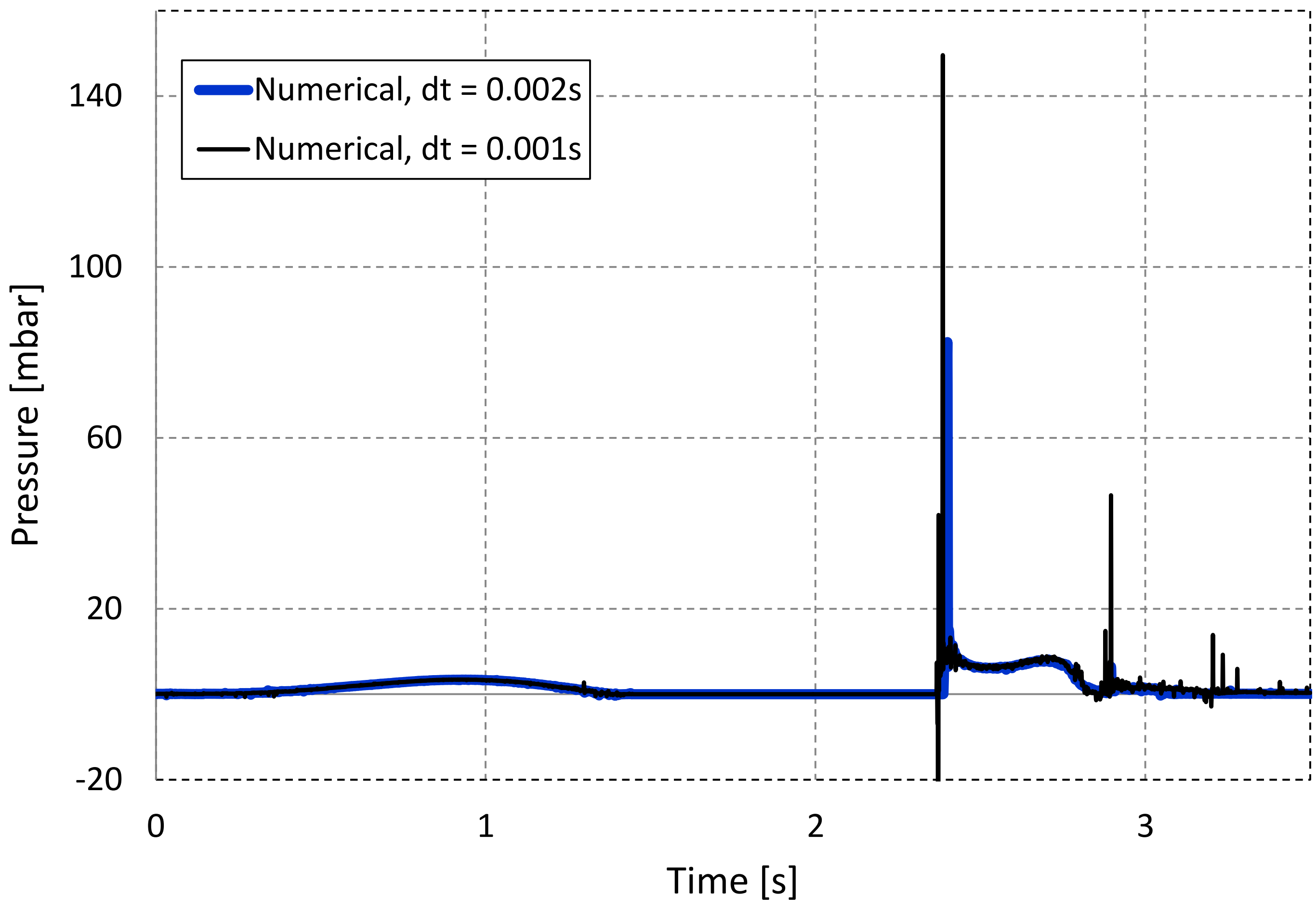
experimental

numerical

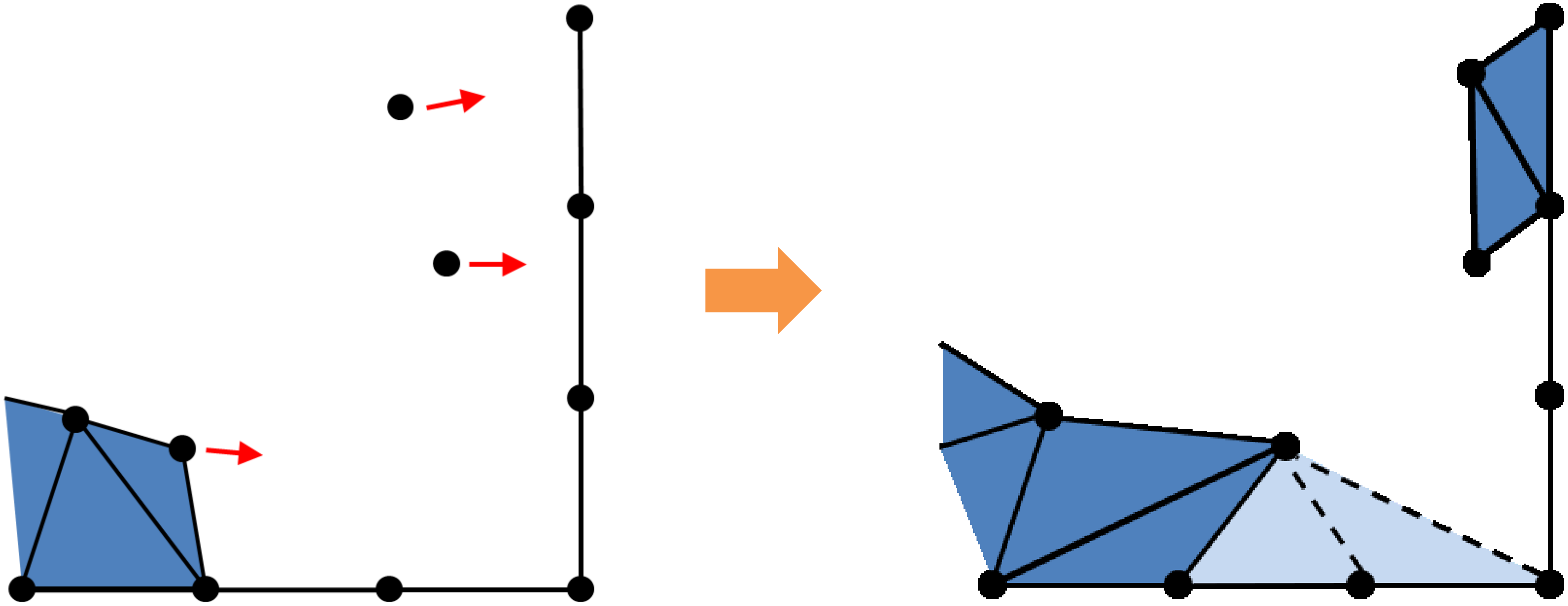
If a reasonable discretization is used pressure evolution appears to be very well reproduced



Nevertheless, pressure oscillations are still present and become visible if the time step is slightly decreased



Pressure oscillations still appear on fluid-solid boundaries due to the way contact is dealt with in the PFEM



Conclusions

Correct free-surface flows formulation:

- Avoid imposing pressure at the free surface
- Do not use so-called «pseudo-tractions»

Remeshing issues:

- Use large time steps (but what about explicit schemes?)
- Use fine discretizations
- Different fluid-solid contact definition

Some references

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